Systems of equations over modules.

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To Professor Alexander Kurosh on his 50th birthday.

§ 1. Introduction.

Let R be an arbitrary ring and G an arbitrary R-module.[1] If x_α $(\alpha \in A)$ is a set of unknowns, then the most general system of equations in these unknowns over G can be written in the form

$$a_{\beta 1} x_{\alpha 1} + \cdots + a_{\beta k_\beta} x_{\alpha k_\beta} + n_{\beta 1} x_{\alpha 1} + \cdots + n_{\beta k_\beta} x_{\alpha k_\beta} = g_\beta \quad (\in G, \beta \in B)$$

with $a_{\beta j} \in R$ and $n_{\beta j}, k_\beta$ rational integers. It is our purpose in this paper to investigate such equation systems over arbitrary operator-modules.

In our investigations the concept of a *compatible system of equations* over a module plays a fundamental role. In § 3 of this paper we also give a "coördinate-free" definition of this concept. By this definition a compatible system of equations over an arbitrary R-module G is a well-defined R-homomorphism φ of a submodule of some free R-module F into G and the solvability of this system is equivalent to the extensibility of the mapping φ to an R-homomorphism $\bar{\varphi}$ of the whole module F into G. This definition turns out to be of a great usefulness in our investigations.

In his paper [18], published in 1950, T. SZELE builds up a theory for abelian groups which is analogous to the Steinitz theory of fields. In this theory the „algebraically closed groups" coincide with the so-called divisible abelian groups.[2] A valuable contribution to the theory of algebraically closed groups was given by S. GACSÁLYI, who succeeded in showing that for an algebraically closed abelian group G any compatible system of equations (containing arbitrarily many unknowns and consisting of an arbitrary set of

[1]) By a ring we shall always mean in this paper an associative ring and by an R-module always a *left* R-module.

[2]) An abelian group G is *divisible* if $G = nG$ for every integer $n (\neq 0)$. For particulars about these groups see e. g. [9]. In conformity with the terminology of SZELE, the divisible abelian groups will be called *algebraically closed* groups in this paper.

equations) over G is solvable in G [7]. In § 4 we generalize for modules with a completely arbitrary ring as domain of operators the theory of algebraically closed abelian groups. It is surprising that even in this most general case many of the results on algebraically closed abelian groups retain their validity. Our method has also the advantage of generalizing some results already known for unitary injective operator-modules, which in this way find their natural place in the frame of a broader theory and by a suitable choice of the logical order also simpler proofs of these results can be obtained.

In § 5 we raise the problem of determining all algebraically closed modules. As a small step towards the solution of this apparently difficult problem we describe all algebraically closed R-modules G for which $RG=0$ holds.

In the last section of our paper (§ 6) we consider the problem of determining all rings R for which any R-module is the direct sum of its maximal trivial submodule[3]) and of an algebraically closed R-module. The solution of this problem leads to the class of semi-simple rings in the classical sense. Moreover, we show that if R is a semi-simple ring and G_1 an arbitrary unitary R-module, then all solutions of any compatible system of equations over G_1 can be obtained with the aid of a suitable set of formulae.

In the special case if the ring R itself being considered as an R-module, our considerations on systems of equations over modules lead to an investigation of systems of linear equations over the ring R. Thus, as a corollary to the results of § 6 we obtain the result that the classical theory of systems of linear equations over skew fields which has been generalized by Gacsályi and Szele for the case of arbitrarily many equations and unknowns carries over to the case when instead of being a skew field the fundamental domain is an arbitrary semi-simple ring. Moreover, the semi-simple rings form the largest class of rings for which the classical theory of linear equations holds.

Some of the results of the present paper have been published without proofs in [12].

§ 2. Preliminaries.

Let R be an arbitrary ring. By an R-module we understand an additively written abelian group which has the ring R as left operator domain. By a submodule resp. by a homomorphism we shall in this paper always mean an R-submodule and an R-homomorphism respectively, i. e. submodu-

[3]) For the terminology see § 2.

les and homomorphisms admissible with respect to the domain of operators. — The ring of rational integers will be denoted by I.

By a *unitary module* we mean a module furnished with an operator domain which is a ring containing a unit element 1 such that 1 acts as the identity operator on the module [2]. Let A be an arbitrary set of elements of the R-module G and let us denote by RA the set of all finite sums with summands of the form ra $(r \in R, a \in A)$. RA is a submodule of G, and of A too, in case A is a submodule of G. If $RG=0$, we say that G is a *trivial R-module*. Any R-module G contains a unique *maximal trivial submodule*, namely the set of all elements $x(\in G)$ such that $Rx=0$.

We denote by $\{S\}$ the submodule of the R-module G generated by the system of elements S of G, i. e. the smallest submodule of G which contains all elements of S. A module generated by a single element is said to be *cyclic*. If $g \in G$, then $\{g\}$ is the set of all elements of the form $rg+ng$ $(r \in R, n \in I)$. If G is a unitary R-module, then the submodule $\{g\}$ consists of the elements of the form rg $(r \in R)$.

By a direct sum of modules we shall always mean a *discrete* direct sum, which will be denoted by $+$ or by Σ, respectively. In some cases we shall also need the „complete" direct sum, which however will always be explicitly termed so. If $G=A+B$, we say that A (and of course also B) is a direct summand of the module G. If A is a submodule of G, such that for any submodule M of G which is maximal with respect to the property $M \cap A=0$ the direct decomposition $G=A+M$ holds, then A will be called a *strictly direct summand* of the module G.[4])

Let x denote an indeterminate element, and H a submodule of the R-module G. Then the most general system of equations which can be written down with this unknown has the form

$$(1) \qquad r_\nu x+n_\nu x=h_\nu \qquad (r_\nu \in R, \ n_\nu \in I, \ h_\nu \in H, \ \nu \in \Delta)$$

where Δ denotes an arbitrary (non void) set of indices. If there exists an element $x(\in G)$ for which the system of equations (1) is fulfilled, then we say that the system of equations (1) is solvable in G and has x as a solution. A submodule H of the R-module G will be called a *pure submodule* in G, if the solvability in G of a system of equations (1) in one unknown always implies its solvability also in H. The fundamental results on pure submodules are given in [11].

So far unitary modules have been given most attention in the literature, and the fundamental concepts of module theory, such as that of the order of

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4) A criterion for a direct summand of an abelian group to be a strictly direct summand was given earlier by L. Fuchs [5].

an element, of a free R-module, of the independence of a system of elements etc. have been established in accordance with this (see e. g. [13]). In the general case however these concepts do not prove satisfactory [5]), and it is necessary to modify them. In what follows we shall make a remark by which it becomes possible to carry over in a natural way to the general case the concepts and methods which have proved so useful in investigating unitary modules.

Let G be an arbitrary R-module. We consider the well-known extension with unit element R^* of the domain of operators R, due to DORROH [3]. The ring R^* consists of all pairs $\langle r, n\rangle$ $(r \in R, n \in I)$, with the operations

$$\left.\begin{aligned}\langle r, n\rangle+\langle s, m\rangle&=\langle r+s, n+m\rangle\\ \langle r, n\rangle\langle s, m\rangle&=\langle rs+mr+ns, nm\rangle\end{aligned}\right\} r, s \in R;\ \ n, m \in I.$$

It is easy to see that in virtue of the definition

$$\langle r, n\rangle g=rg+ng \qquad (g \in G)$$

the module G becomes an R^*-module, and indeed a unitary R^*-module, since for the unit element $\langle 0,1\rangle$ of the ring R^*

$$\langle 0,1\rangle g=0\cdot g+1\cdot g=g$$

holds with any element $g(\in G)$.

Now it is clear that if the structure of the module G is being envisaged, this may be considered indifferently as an R-module or as an R^*-module, and thus, in investigating many problems concerning operator-modules we may restrict ourselves without loss of generality to the case of unitary modules. Besides, in introducing the necessary concepts in the general theory of operator-modules, it is covenient to consider the arbitrary R-module G in the above manner as an R^*-module, and so to give the definitions to be used in the case of unitary modules. In the sequel only the concept of a free R-module will be defined in such a way, for this is the only concept we shall need.

We stipulate that for an arbitrary ring R, R^* shall always denote in the present paper the Dorroh extension with unit element of R, and that if we consider the R-module G as an R^*-module, this shall always mean that the ring R^* corresponds to the module G in the above standard way as a domain of operators. To be more explicit, for a given R-module G, the product

$$\langle r, n\rangle g \qquad (\langle r, n\rangle \in R^*, g \in G)$$

[5]) Let us mention but one example for this. In the literature (see e. g. [9] and [13]) an R-module is called free if it is a direct sum of copies of R considered as an R-module. If, however R is a ring with unit element, then only the unitary R-modules can be obtained as factor modules of such free R-modules.

shall always denote in the sequel the element $rg+ng(\in G)$. If, however, in some particular context the ring R is a priori known to have a unit element and all R-modules considered are unitary ones, the introduction of the domain of operators R^* is, of course, completely superfluous.

Let R be an arbitrary ring and S a system of symbols x_α, where α runs through an index-set A of cardinality $\mathfrak{m}(>0)$. Consider the set F of formal sums

$$f=\sum_{i=1}^{k}\langle r_i, n_i\rangle x_{\alpha_i} \qquad (\langle r_i, n_i\rangle\in R^*)$$

(with a finite number of terms), the indices α_i $(i=1,\ldots,k)$ being pairwise different. An expression of this form will be called a *linear form over* R. If for all i's $\langle r_i, n_i\rangle=\langle 0,0\rangle$ holds, we write $f=0$. Two linear forms

$$f_1=\sum_{i=1}^{k_1}\langle r_{1i}, n_{1i}\rangle x_{\alpha_{1i}} \quad \text{and} \quad f_2=\sum_{j=1}^{k_2}\langle r_{2j}, n_{2j}\rangle x_{\alpha_{2j}}$$

will be considered equal, if for any pair of indices i,j, for which $\alpha_{1i}=\alpha_{2j}$ holds, one has the equality $\langle r_{1i}, n_{1i}\rangle=\langle r_{2j}, n_{2j}\rangle$ and for indices i to which no index j with $\alpha_{1i}=\alpha_{2j}$ corresponds one has $\langle r_{1i}, n_{1i}\rangle=\langle 0,0\rangle$, and similarly for the j's. The set F becomes an R-module by introducing the operations

$$sf=\sum_{i=1}^{k}\langle sr_i+n_is, 0\rangle x_{\alpha_i} \qquad (s\in R)$$

and

$$f_1+f_2=\sum_{i=1}^{k_1}\langle r_{1i}, n_{1i}\rangle x_{\alpha_{1i}}+\sum_{j=1}^{k_2}\langle r_{2j}, n_{2j}\rangle x_{\alpha_{2j}}, \tag{2}$$

the expression (2) being converted into a linear form by carrying out all possible „simplifications". (More explicitly, if for some pair of indices i,j the equality $\alpha_{1i}=\alpha_{2j}$ holds, then the sum $\langle r_{1i}, n_{1i}\rangle x_{\alpha_{1i}}+\langle r_{2j}, n_{2j}\rangle x_{\alpha_{2j}}$ must be replaced in (2) by $\langle r_{1i}+r_{2j}, n_{1i}+n_{2j}\rangle x_{\alpha_{1i}}$.)

The R-module F so obtained will be called a *free R-module*. Since the free R-module F is essentially uniquely determined by the ring R and by the cardinal number $\mathfrak{m}$, we shall denote it by the symbol $R(\mathfrak{m})$. Making the identification $\langle 0,1\rangle x_\alpha=x_\alpha$ $(\alpha\in A)$ we obtain $x_\alpha\in F$ and thus the system S of the elements x_α $(\alpha\in A)$ generates the module F. The system of elements S will be called a *free base* of the module F.

Let R_1 be a ring with a unit element 1, F_1 a unitary R-module and X a subset of F_1. We shall say that F_1 is a *free unitary R_1-module* with X as a *free base* if every element f of F_1 can be written uniquely as a finite sum Σr_ix_i $(r_i\in R, x_i\in X)$. It is easy to see that F_1 is isomorphic to a direct sum of R-modules, any of which is isomorphic to the ring R considered as an R-module.[6])

Finally we list the more important notations used throughout this paper:

I: the ring of rational integers;
R^*: the Dorroh extension with unit element of the ring R;
$L_{(R)}$: if R is a subring of the ring S and L a left ideal of R, then we denote by $L_{(R)}$ the additive group of L considered as R-module, where the product rl ($r \in R$, $l \in L$) coincides with the product rl defined in S;[7])
G/H: the factor module of G with respect to H;
$+, \Sigma$: for elements their sum, for modules their direct sum;
$\oplus$: the ring-theoretical direct sum of rings;
$H_1 \subseteq H_2$: H_1 is a subset of H_2;
$H_1 \subset H_2$: H_1 is a proper subset of H_2;
$\{\ldots, f_\beta, \ldots\}_{\beta \in B}$: the submodule generated by all elements f_β ($\beta \in B$);
$R(\mathfrak{m})$: the free R-module generated by a free base of cardinality $\mathfrak{m}$.

§ 3. Compatible systems of equations over modules.

Let R be an arbitrary ring, G an arbitrary R-module and x an unknown. The most general equation over G in this unknown has the form

$$rx + nx = g \qquad (r \in R;\ n \in I;\ g \in G). \tag{3}$$

If there exists an element $x (\in G)$ for which the equation (3) holds, we say that the equation (3) is solvable in G and has x as a solution.

Let us consider the following example: Let $R = I \oplus I$. (We denote the elements of the ring R by pairs of elements (n, m).) The equation $(0,1)(x, y) = (1,0)$ over $R_{(R)}$ is not solvable in $R_{(R)}$, nor in any extension of $R_{(R)}$,[8]) for if we multiply both sides of this equation e. g. by $(1,0)$, we obtain zero on the left-hand side, and $(1,0)$, which is different from zero, on the right-hand side. This example shows how essential the question of compatibility

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6) We remark that the concept of a *free R-module* is not a generalization, but only an analogue of the concept of a *free unitary R-module*. The two concepts are in the same relation as e. g. „free group" and „free abelian group".

7) More exactly, we ought to introduce a notation which would express that we are dealing with a substructure of S. This however would make the notation very cumbersome, and in the cases considered the simpler notation above can be used without fear of confusion.

8) I. e. in an R-module having $R_{(R)}$ as a submodule.

is even in the case of a single equation in one unknown. In order to be able to investigate equations over modules, we must first of all define the concept of a compatible system of equations.

Let R be an arbitrary ring, A an arbitrary set of indices of cardinality $\mathfrak{m}\,(>0)$ and S the system of the symbols x_α $(\alpha\in A)$. Let us consider, over an R-module G, the system of equations

(4) $$f_\beta = g_\beta \qquad (\beta\in B,\ g_\beta\in G)$$

where B is an arbitrary (non-void) set of indices and f_β an element of the free R-module $R(\mathfrak{m})$ having the free base x_α $(\alpha\in A)$, i. e.

(5) $$f_\beta = \langle a_{\beta 1}, n_{\beta 1}\rangle x_{\alpha 1} + \cdots + \langle a_{\beta k_\beta}, n_{\beta k_\beta}\rangle x_{\alpha k_\beta} \qquad (\langle a_{\beta j}, n_{\beta j}\rangle \in R^*)$$

is a linear form over R^*. We say that the system of elements

(6) $$x_\alpha = h_\alpha\ (\in G,\ \alpha\in A)$$

is a solution of the system of equations (4) in G, if (6) satisfies all equations of the system (4). Obviously, the following is a trivial necessary condition for the solvability of the system (4): every relation for a finite number of f_β's, obtained by a linear combination over R^*, must be satisfied also by the corresponding constants g_β on the right-hand sides of our system of equations. In other words, every relation of the form

$$\langle s_1, m_1\rangle f_{\beta_1} + \cdots + \langle s_i, m_i\rangle f_{\beta_i} = 0$$

must imply

$$\langle s_1, m_1\rangle g_{\beta_1} + \cdots + \langle s_i, m_i\rangle g_{\beta_i} = 0.$$

We call this requirement the *condition of compatibility.*

Let now be (4) a system of equations satisfying the condition of compatibility. Now the system of equations (4) yields a well-defined R-homomorphism φ into G of the submodule M of the free R-module $R(\mathfrak{m})$ generated by all linear forms f_β $(\beta\in B)$, as follows:

$$(\langle t_1, l_1\rangle f_{\beta_1} + \cdots + \langle t_k, l_k\rangle f_{\beta_k})\,\varphi = \langle t_1, l_1\rangle g_{\beta_1} + \cdots + \langle t_k, l_k\rangle g_{\beta_k}.$$

Then in particular $f_\beta\varphi = g_\beta$ $(\beta\in B)$ holds. (The mapping is single-valued: this is assured by the compatibility of the system (4).) — Conversely: a given R-homomorphism φ of a submodule M of $R(\mathfrak{m})$ into G yields always compatible systems (4) of equations over G; in any of these systems the left-hand sides f_β form a generating system of M and the corresponding right-hand sides are by $b_\beta = f_\beta\varphi$ defined. — If two systems of equations over G satisfying the condition of compatibility spring in the above way from the same R-homomorphism φ of the same submodule M of $R(\mathfrak{m})$, then we call these systems *equivalent.* Obviously, two systems of equations over G are equivelent if and only if each equation of the one system can be obtained as a (left-)linear combination over R^* of a finite number of equations of the other system, and conversely. Also it is clear that the solutions of two equivalent systems of equations (satisfying the condition of compatibility) coincide. Since equivalent systems of equations can thus be considered as being essentially the same, we are led to the following

Definition. *The compatible system $[M, \varphi]$ of equations in $\mathfrak{m}$ unknowns over an arbitrary R-module G is the given R-homomorphism φ of the submodule M of the free R-module $R(\mathfrak{m})$ into G. The system $[M, \varphi]$ of equations is called homogeneous, if $M\varphi = 0$.*

Making use of a valuable remark by G. Pollák, we are able to complete our definition in the following way:

The compatible system $[M, \varphi]$ of equations over G is solvable in G if and only if the R-homomorphism φ of the submodule M can be extended to an R-homomorphism $\overline{\varphi}$ of the whole free R-module $R(\mathfrak{m})$ into G. The solutions of the system $[M, \varphi]$ of equations are in one-to-one correspondence with the extensions $\overline{\varphi}$ of the homomorphism φ, and so by a solution of the system $[M, \varphi]$ of equations we always mean the corresponding extension $\overline{\varphi}$ of the homomorphism φ.

As a matter of fact, let us suppose that the system of elements (6) is a solution of the system of equations (4). The mapping $x_\alpha \to h_\alpha$ $(\alpha \in A)$ induces a homomorphism $\overline{\varphi}$ of the whole module $R(\mathfrak{m})$ into G. Now

$$f_\beta\overline{\varphi} = \langle a_{\beta 1}, n_{\beta 1}\rangle (x_{\alpha 1}\overline{\varphi}) + \cdots + \langle a_{\beta k_\beta}, n_{\beta k_\beta}\rangle (x_{\alpha k_\beta}\overline{\varphi}) =$$
$$= \langle a_{\beta 1}, n_{\beta 1}\rangle h_{\alpha 1} + \cdots + \langle a_{\beta k_\beta}, n_{\beta k_\beta}\rangle h_{\alpha k_\beta} = g_\beta = f_\beta \varphi$$

and thus $\overline{\varphi}$ is a suitable extension of the mapping φ. — Conversely, suppose that the R-homomorphism φ induced by the system of equations (4) can be extended to an R-homomorphism $\overline{\varphi}$ of the whole module $R(\mathfrak{m})$ into G. Let in particular be $x_\alpha\overline{\varphi} = h_\alpha$ $(\alpha \in A)$. Then

$$f_\beta\overline{\varphi} = (\langle a_{\beta 1}, n_{\beta 1}\rangle x_{\alpha 1} + \cdots + \langle a_{\beta k_\beta}, n_{\beta k_\beta}\rangle x_{\alpha k_\beta})\overline{\varphi} =$$
$$= \langle a_{\beta 1}, n_{\beta 1}\rangle (x_{\alpha 1}\overline{\varphi}) + \cdots + \langle a_{\beta k_\beta}, n_{\beta k_\beta}\rangle (x_{\alpha k_\beta}\overline{\varphi}) =$$
$$= \langle a_{\beta 1}, n_{\beta 1}\rangle h_{\alpha 1} + \cdots + \langle a_{\beta k_\beta}, n_{\beta k_\beta}\rangle h_{\alpha k_\beta}.$$

On the other hand, since on M the mapping $\overline{\varphi}$ coincides with φ, we have by the definition of φ

$$f_\beta\overline{\varphi} = f_\beta\varphi = g_\beta.$$

Thus we have shown that $x_\alpha = h_\alpha$ $(\alpha \in A)$ is a solution of the system (4) of equations in G.

Let us consider an *arbitrary* system of equations

$$(7) \qquad f_\beta = g_\beta \qquad (f_\beta \in R(\mathfrak{m}),\ g_\beta \in G,\ \beta \in B)$$

over the R-module G. We denote by $G(\mathfrak{m})$ the direct sum of the modules G and $R(\mathfrak{m})$:

(8) $$G(\mathfrak{m}) = G + R(\mathfrak{m}).$$

Then we have the following

Theorem 1. *For the system* (7) *of equations over the R-module G the following conditions are equivalent:*

α) *the system* (7) *of equations is compatible;*

β) *the set of all elements $f_\beta - g_\beta$ ($\beta \in B$) generates in the module* $G(\mathfrak{m})$ *a submodule H, such that $G \cap H = 0$;*

γ) *the system* (7) *of equations is solvable in some extension of G.*[9])

Proof. α) *implies* β). Suppose

$$\langle r_1, n_1\rangle(f_{\beta_1} - g_{\beta_1}) + \cdots + \langle r_j, n_j\rangle(f_{\beta_j} - g_{\beta_j}) = g \in G.$$

This implies

$$(\langle r_1, n_1\rangle f_{\beta_1} + \cdots + \langle r_j, n_j\rangle f_{\beta_j}) - (\langle r_1, n_1\rangle g_{\beta_1} + \cdots + \langle r_j, n_j\rangle g_{\beta_j}) = g.$$

In view of the direct decompositions (8), $\langle r_1, n_1\rangle f_{\beta_1} + \cdots + \langle r_j, n_j\rangle f_{\beta_j} = 0$, and thus α) implies $\langle r_1, n_1\rangle g_{\beta_1} + \cdots + \langle r_j, n_j\rangle g_{\beta_j} = 0$ and therefore $g = 0$.

β) *implies* γ). Consider the factor module $\overline{G(\mathfrak{m})} = G(\mathfrak{m})/H$. In this factor module the elements $\bar{g}$ ($g \in G$) form, in view of condition β), an R-module isomorphic to G, and therefore $\overline{G(\mathfrak{m})}$ may be considered as an extension of G. The mapping $f \to \bar{f}$ ($f \in R(\mathfrak{m})$) is an R-homomorphism of the module $R(\mathfrak{m})$ into $\overline{G(\mathfrak{m})}$, which extends the R-homomorphism induced by the system of equations

$$f_\beta = \bar{g}_\beta \qquad (\beta \in B)$$

to the whole of $R(\mathfrak{m})$. Consequently, the system (7) of equations admits a solution in $\overline{G(\mathfrak{m})}$.

γ) *implies* α). Let $\bar{\varphi}$ be an R-homomorphism of $R(\mathfrak{m})$ into some extension G' of G for which $f_\beta \bar{\varphi} = g_\beta$ ($\beta \in B$) holds. The existence of such a homomorphism $\bar{\varphi}$ is assured by γ). Then it is clear that the mapping φ into G of the submodule $M = \{\ldots, f_\beta, \ldots\}_{\beta \in B}$ of the module $R(\mathfrak{m})$ induced by the mapping $f_\beta \to g_\beta$ ($\beta \in B$) concides with $\bar{\varphi}$ on the submodule M, i. e. it is an R-homomorphism. This proves the validity of property α), completing at the same time the proof of the theorem.

[9]) The equivalence of the ring-theoretical analoga of the conditions β) and γ) is established in [17] and [20].

§ 4. Algebraically closed modules.

Let R be an arbitrary ring and G an R-module. The module G is called *algebraically closed*, if every compatible system of equations in one unknown over G is solvable in G.

First of all we remark that in this definition the condition of the solvability of any compatible *system* of equations in one unknown cannot be replaced by the condition of the solvability of any compatible (single) *equation* in one unknown. This is shown by the following example:

Let K be the complete direct sum of an infinite number of fields K_ν $(\nu \in N)$. The discrete direct sum K_0 of the fields K_ν is an ideal in K, so that K_0 can be regarded as a K-module. The ring K is a regular ring with unit element in the sense of J. VON NEUMANN,[10]) and so, by a theorem of NEUMANN (see [15], vol. 2, ch. 2.) any principal ideal of K is a direct summand. From this it follows that any compatible (single) *equation* in one unknown over the K-module K_0 is solvable in K_0. On the other hand, let us consider the *system* of equations consisting of all equations $k_\beta x = k_\beta$ $(k_\beta \in K_0)$. This system is obviously compatible, but it is not solvable in K_0, for K_0 has no unit element.

Several important characteristic properties of algebraically closed modules are expressed by the following

Theorem 2.[11]) *For an arbitrary R-module G the following conditions are equivalent:*

a) *G is algebraically closed;*

b) *if G is a direct summand of the R-module D, then it is a strictly direct summand of D;*

c) *to every left ideal $L_{(R)}$ of R^* and to every R-homomorphism ψ of $L_{(R)}$ into G there exists some element $g_0 (\in G)$ so that $\langle s, m\rangle\psi = \langle s, m\rangle g_0$ for every element $\langle s, m\rangle$ in $L_{(R)}$;*

d) *if φ is a homomorphism of some submodule A of an arbitrary R-module B into G, then φ can be extended to a homomorphism of the whole B into G;*

e) *every compatible system of equations over G admits a solution in G;*

f) *if G is a submodule of the R-module D, then it is a direct summand of D;*

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[10]) A ring R is called *regular* if for each element $r (\in R)$ there exists an element $x (\in R)$ such that $rxr = r$ [14].

[11]) This theorem shows that in the special case of unitary modules the algebraically closed modules coincide with the *complete modules* introduced by R. BAER and with the *injective modules* of homological algebra. The results of this section generalize some results already known for unitary injective modules. (See [1] and [4].)

g) *if G is a submodule of the R-module D, then it is a pure submodule in D.*

As immediate consequences of this theorem we have the following corollaries:

Corollary 1. *If any compatible system of linear equations in a single unknown over the ring R is solvable in R, then any compatible system of linear equations (with arbitrarily many unknowns and equations) over R is also solvable in R.*

Corollary 2. (S. Gacsályi [8].) *Any compatible system of linear equations over a skew field F possesses a solution in F.*

Proof of the theorem. a) *implies* b).[12]) Let G be a submodule of the R-module D and H another submodule of D, which is maximal with respect to the property $G \cap H = 0$. (The existence of such a H is assured by Zorn's lemma.) We show that

$$D = G + H. \tag{9}$$

Adopting the notation $K = G + H$, let us suppose that $K \subset D$. Then there exists an element $d (\in D)$ for which $d \notin K$. The set of all elements $\langle s_\nu, m_\nu \rangle$ $(\in R^*)$ for which $\langle s_\nu, m_\nu \rangle d \in K$ is a left ideal L of the ring R^*. (Because of the maximality of H, $L \neq 0$.) By the construction of K there exist for each element $\langle s_\nu, m_\nu \rangle$ $(\in L)$ uniquely determined elements g_ν and h_ν in G and H respectively, so that

$$\langle s_\nu, m_\nu \rangle d = g_\nu + h_\nu .$$

Obviously the system of equations

$$\langle s_\nu, m_\nu \rangle x = g_\nu$$

is compatible, and therefore by condition a) there exists in G an element g_0 so that

$$\langle s_\nu, m_\nu \rangle g_0 = g_\nu$$

for every $\langle s_\nu, m_\nu \rangle$ in L. Let us consider the element $d' = d - g_0$. This element satisfies the relations $d' \notin K$,

$$\langle s_\nu, m_\nu \rangle d' = h_\nu \in H \quad \text{if} \quad \langle s_\nu, m_\nu \rangle \in L \tag{10}$$

and

$$\langle r, n \rangle d' \notin K \quad \text{if} \quad \langle r, n \rangle \notin L.$$

Since H is a greatest submodule of D for which $G \cap H = 0$ and since $d' \notin K$,

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12) We are, in fact, proving more than what has been asserted: by the well-known method of Baer (see [1]) we show that if G is a submodule of D, then it is also a strictly direct summand of D.

we have $\{H, d'\} \cap G \neq 0$. Therefore there exist h_0 $(\in H)$ and $\langle r_0, n_0 \rangle$ $(\in R^*)$ such that

$$h_0 + \langle r_0, n_0 \rangle d' = g \neq 0 \qquad (g \in G).$$

Since $\langle r_0, n_0 \rangle d' = g - h_0 \in K$, it follows that $\langle r_0, n_0 \rangle \in L$ and so by (10) $\langle r_0, n_0 \rangle d' \in H$ i. e. $g \in H$. This however contradicts the construction of H and the choice of g. This contradiction proves the stated decomposition (9) of D.

b) *implies* c). Let ψ be an R-homomorphism into G of some left ideal $L_{(B)}$ of the ring R^*, put

$$D = G + R(1), \tag{11}$$

and let x be some free generating element of the free R-module $R(1)$. Then the set of all elements $\langle s, m \rangle \psi - \langle s, m \rangle x$ $(\langle s, m \rangle \in L_{(B)})$ is a submodule M_0 of D, for which $M_0 \cap G = 0$ holds. (By the decomposition (11) the relation $\langle s, m \rangle \psi - \langle s, m \rangle x = g$ $(\in G)$ evidently implies $g = 0$.) Let now M be a submodule of D, maximal with respect to the properties $M_0 \subseteq M$ and $M \cap G = 0$. By condition b),

$$D = G + M, \tag{12}$$

and so there exist uniquely determined elements $g_1 (\in G)$ and $g_2 (\in M)$ for which $x = g_1 + g_2$. This yields for all elements $\langle s, m \rangle (\in L_{(B)})$

$$\langle s, m \rangle x = \langle s, m \rangle g_1 + \langle s, m \rangle g_2. \tag{13}$$

On the other hand

$$\langle s, m \rangle x = \langle s, m \rangle \psi - (\langle s, m \rangle \psi - \langle s, m \rangle x), \tag{14}$$

with

$$\langle s, m \rangle \psi - \langle s, m \rangle x \in M_0.$$

Since the components of the element $\langle s, m \rangle x$ in the decomposition (12) are uniquely determined, we obtain by (13) and (14)

$$\langle s, m \rangle \psi = \langle s, m \rangle g_1$$

for any $\langle s, m \rangle$ $(\in L_{(B)})$, and this proves our assertion.

c) *implies* d). Let φ be a homomorphism of some submodule A of an arbitrary R-module B into G. Let us suppose, moreover, that condition c) is satisfied. Then we have the following

Lemma 1. *If b is an arbitrary element of B, then φ can always be extended to a homomorphic mapping of the module $\{A, b\}$ into G.*

Proof. 1. If $\{b\} \cap A = 0$, then $\{A, b\} = A + \{b\}$, and the possibility of extending φ is evident.

2. Let $\{b\} \cap A \neq 0$ and let L be the left ideal generated by all those elements $\langle s, m\rangle$ of the ring R^* for which $\langle s, m\rangle b \in A$ holds. The mapping

$$\langle s, m\rangle \to (\langle s, m\rangle b)\varphi \qquad (\langle s, m\rangle \in L)$$

is an R-homomorphism of $L_{(R)}$ into G, and consequently there exists an element $g_0 (\in G)$, such that

$$(\langle s, m\rangle b)\varphi = \langle s, m\rangle g_0 \tag{15}$$

holds for any element $\langle s, m\rangle$ $(\in L)$. We show that the mapping

$$a + \langle r, n\rangle b \to a\varphi + \langle r, n\rangle g_0 \qquad (a \in A;\ \langle r, n\rangle \in R^*) \tag{16}$$

is an R-homomorphism of the module $\{A, b\}$ into G, representing an extension of the homomorphism φ. One clearly has $a \to a\varphi$ for any element a $(\in A)$, and on the other hand the image of the sum of two elements is equal to the sum of the images of the two elements. Thus we have only to show that the mapping is single-valued. Suppose an equality of the form

$$a_1 + \langle r_1, n_1\rangle b = a_2 + \langle r_2, n_2\rangle b \qquad (a_1, a_2 \in A;\ \langle r_1, n_1\rangle, \langle r_2, n_2\rangle \in R^*) \tag{17}$$

holds. From this

$$(\langle r_2, n_2\rangle - \langle r_1, n_1\rangle) b = a_1 - a_2 \in A,$$

and consequently

$$\langle r_2, n_2\rangle - \langle r_1, n_1\rangle = \langle s_0, m_0\rangle \in L \tag{18}$$

follows. By the mapping (16)

$$a_1 + \langle r_1, n_1\rangle b \to a_1\varphi + \langle r_1, n_1\rangle g_0$$

and

$$a_2 + \langle r_2, n_2\rangle b \to a_2\varphi + \langle r_2, n_2\rangle g_0.$$

We show that $a_1\varphi + \langle r_1, n_1\rangle g_0 = a_2\varphi + \langle r_2, n_2\rangle g_0$. Indeed, by the equalities (17), (18) and (15) we have

$$\begin{aligned} a_1\varphi + \langle r_1, n_1\rangle g_0 &= (a_2 + \langle r_2, n_2\rangle b - \langle r_1, n_1\rangle b)\varphi + \langle r_1, n_1\rangle g_0 = \\ &= a_2\varphi + (\langle s_0, m_0\rangle b)\varphi + \langle r_1, n_1\rangle g_0 = a_2\varphi + \langle s_0, m_0\rangle g_0 + \\ &\quad + \langle r_1, n_1\rangle g_0 = a_2\varphi + \langle r_2, n_2\rangle g_0. \end{aligned}$$

This completes the proof of Lemma 1.

We are going to prove the implication c) $\to$ d) with the aid of Lemma 1:

Let us consider the (evidently non-void) set of all pairs of elements (H_μ, φ_μ), for which H_μ is a submodule containing the submodule A of B, and φ_μ a homomorphism of H_μ into G extending the homomorphism φ. This set is partially ordered with respect to the following relation: $(H_\lambda, \varphi_\lambda) \leqq (H_\mu, \varphi_\mu)$ if $H_\lambda \subseteq H_\mu$ and φ_μ is an extension of φ_λ. Since this partially ordered set is inductive, there exists by Zorn's lemma a maximal pair of elements (H_0, φ_0). Now, by our lemma, H_0 must coincide with B, and thus φ_0 is a homomorphism extending the homomorphism φ of the whole module B into G.

d) *implies* e). As a matter of fact, if d) holds for any B, then, in particular, it holds also in the case when B is a free R-module, and consequently any compatible system of equations over G is solvable in G.[13])

e) *implies* f). In the proof we make use of a part of the assertion of the following lemma, which is of some interest also for its own sake:

Lemma 2. *A submodule A of an arbitrary R-module H is a direct summand of H if and only if any system of equations over A, solvable in H, is solvable also in A.*[14])

Proof. Suppose that

$$H=A+B \tag{19}$$

and that the system of equations

$$f_\beta(\ldots,x_\nu,\ldots)=c_\beta \ (\in A) \tag{20}$$

is solvable in H. If $\ldots,h_\nu,\ldots$ $(\in H)$ is a solution of the system (20), then from the unique decomposition

$$h_\nu=a_\nu+b_\nu \qquad (a_\nu\in A, b_\nu\in B)$$

implied by (19) we obtain

$$f_\beta(\ldots,h_\nu,\ldots)=f_\beta(\ldots,a_\nu,\ldots)+f_\beta(\ldots,b_\nu,\ldots)=c_\beta \ (\in A),$$

and so

$$f_\beta(\ldots,a_\nu,\ldots)=c_\beta.$$

Thus the system of elements $\ldots,a_\nu,\ldots$ is a solution in A of the system of equations (20).

Suppose that, conversely, any system of equations over A, solvable in H, is solvable also in A. Let $\ldots,x_\mu,\ldots$ be a system of elements of H, such that

$$H=\{A,\ldots,x_\mu,\ldots\}. \tag{21}$$

---

13) Here is still another proof of the implication d)→e). Let

$$f_\beta(\ldots,x_\alpha,\ldots)=g_\beta \tag{$*$}$$

be an arbitrary compatible system of equations over G. Then, by Theorem 1, there exists an R-module K, which has G as a submodule and in which the system of equations ($*$) is solvable. Let $\ldots,x_\alpha,\ldots$ $(\in K)$ be an arbitrary solution. Since $g\varphi=g$ is a homomorphic mapping of the submodule G of the module K onto G, it can be extended by d) to a homomorphism $\overline{\varphi}$ of the whole of K onto G. Then

$$f_\beta(\ldots,x_\alpha\overline{\varphi},\ldots)=[f_\beta(\ldots,x_\alpha,\ldots)]\overline{\varphi}=g_\beta\overline{\varphi}=g_\beta\varphi=g_\beta,$$

i. e. the system of elements $\ldots,(x_\alpha\overline{\varphi}),\ldots$ $(\in G)$ is a solution in G of the system of equations ($*$).

14) In the case of ordinary abelian groups this lemma is due to Gacsályi [8]. The proof too is a suitable modification of that given by Gacsályi.

Now consider all valid relations of the form

(22) $$\langle r_1, n_1\rangle x_{\mu_1} + \cdots + \langle r_k, n_k\rangle x_{\mu_k} = a \, (\in A;\ \langle r_i, n_i\rangle \in R^*).$$

The system of these relations may be considered as a system of equations over A, having the system of elements $\ldots, x_\mu, \ldots$ as a solution. By our hypothesis this system of equations is satisfied also by some system $\ldots, y_\mu, \ldots$ of elements of A, i. e.

$$\langle r_1, n_1\rangle y_{\mu_1} + \cdots + \langle r_k, n_k\rangle y_{\mu_k} = a.$$

We show that

(23) $$H = A + B$$

with

$$B = \{\ldots, x_\mu - y_\mu, \ldots\}.$$

As a matter of fact, the modules A and B together generate the whole module H, since $\{A, B\}$ contains all elements x_μ and (21) holds. On the other hand let

$$\langle r_1, n_1\rangle (x_{\mu_1} - y_{\mu_1}) + \cdots + \langle r_k, n_k\rangle (x_{\mu_k} - y_{\mu_k}) = a' \ (\in A)$$

be an arbitrary element of the intersection $A \cap B$. This is one of the relations (22), satisfied, by our hypothesis, also for $\ldots, x_\mu = y_\mu, \ldots$ so that $a' = 0$, i. e. $A \cap B = 0$. This proves the relation (23) and thus also Lemma 2.

Let us now suppose that G is a submodule of the R-module D. Since any system of equations over G solvable in D is compatible and so by e) must have a solution in G, we can apply Lemma 2, which shows that G is a direct summand of the module D.

f) *implies* g). This is clear, since a direct summand is always a pure submodule.

g) *implies* a). Let

(24) $$f_\beta(x) = g_\beta$$

be a compatible system of equations in one unknown over G. Then, by Theorem 1, there exists an R-module K, having G as a submodule, in which the system of equations (24) is solvable. Since by g) G is a pure submodule in K, the system of equations (24) is solvable also in G, so that G is an algebraically closed R-module.

This completes the proof of Theorem 2.

If the R-module A is a submodule of the R-module K, we say that K is an *extension of* A. If $A \subset K$, we speak of a proper extension. Let $0 \neq x \in K$. We say that the element x is *algebraic over* A, if it is the solution of some equation $\langle r, n\rangle x = a$ ($\langle r, n\rangle \in R^*$; $0 \neq a \in A$) over A. In the contrary case we say that x is a *transcendental element over* A. The module K is an *algebraic* *extension* of A, if every nonzero element of K is algebraic over A. Otherwise we say that K is a *transcendental extension* of A.

Lemma 3. *Let the R-module K be an extension of the R-module A. The module K is an algebraic extension of A if and only if for any submodule H of K the relation $H \cap A = 0$ implies $H = 0$.*

Proof. Let us suppose that K is an algebraic extension of A, and that $H \subset K$ and $A \cap H = 0$. Then for any element $(0 \neq) h$ $(\in H)$ there exists an $\langle r, n \rangle$ $(\in R^*)$ such that $0 \neq \langle r, n \rangle h \in A$. Since, on the other hand, $\langle r, n \rangle h \in H$ we must have $H = 0$. — Conversely, if for any submodule H of K, $H \cap A = 0$ implies $H = 0$, then for the element $(0 \neq) x$ $(\in K)$ the relation $\{x\} \cap A \neq 0$ holds, and so there exists an $\langle r, n \rangle$ $(\in R^*)$ such that $\langle r, n \rangle x = a$ $(\neq 0, a \in A)$ is fulfilled.

Lemma 4. *Let A $(\neq 0)$ and B be submodules of the R-module K. If $A \cap B = 0$ and B is maximal with respect to this property, then the module K/B is an algebraic extension of the module $(A+B)/B$.*

Proof. Let indeed $k + B$ $(k \notin B)$ be an arbitrary nonzero element of K/B. Then, by the choice of B, one has $(\{k, B\} \cap (A+B)) \subseteq\mathrel{\mkern-3mu}\neq B$, so that $k+B$ is an element of K/B algebraic over $(A+B)/B$.

Theorem 3. *An R-module A is algebraically closed if and only if it has no proper algebraic extensions.*

Proof. Let K be an extension of the algebraically closed module A. Then, by Theorem 2 there exists a submodule H of K such that $K = A \dotplus H$. If K is a proper extension, then $H \neq 0$ and a nonzero element of H cannot be algebraic over A. — Conversely, let us suppose that A has no proper algebraic extension, and let K be an arbitrary submodule containing A. We show that A is a direct summand of K, which, by Theorem 2, amounts to proving our assertion. If $A = 0$, we have nothing to show. Therefore we suppose $A \neq 0$. Let H be a submodule of K which is maximal with respect to the property $A \cap H = 0$. Then, by Lemma 4, K/H is an *algebraic* extension of the module $(A+H)/H$ which is isomorphic to A, and by our hypothesis this is only possible if $A + H = K$.

Let K be an arbitrary extension of the R-module A. We call an arbitrary finite set $h_1, \ldots, h_k$ of nonzero elements in K *algebraically independent over A*, if a relation

$$\langle r_1, n_1 \rangle h_1 + \cdots + \langle r_k, n_k \rangle h_k = a \quad (\in A)$$

always implies

$$\langle r_1, n_1 \rangle h_1 = \cdots = \langle r_k, n_k \rangle h_k = 0.$$

A system of arbitrary cardinality of elements of K is algebraically independent over A, if any of its finite subsystems has this property.[15]) We say that K is a *purely transcendental extension* of A, if the module K is generated by A and by a system S of elements algebraically independent over A.

Theorem 4. *Any extension K of an arbitrary R-module A can be obtained as the result of a purely transcendental extension followed by an algebraic extension.*

Proof. Let S be a system of elements of K, maximal algebraically independent over A. (The existence of such a system of elements S follows from Zorn's lemma.) Then $\{A, S\}$ is a purely transcendental extension of A. Let be $(0 \neq)\ g \in K$ and $g \notin \{A, S\}$. By the maximality of the system S a relation of the form

$$\langle r, n\rangle g + \langle r_1, n_1\rangle h_1 + \cdots + \langle r_k, n_k\rangle h_k = a \ \ (\in A)$$

holds, with $h_1, \ldots, h_k \in S$ and $\langle r, n\rangle g \neq 0$. Thus

$$0 \neq \langle r, n\rangle g = a - (\langle r_1, n_1\rangle h_1 + \cdots + \langle r_k, n_k\rangle h_k) \in \{A, S\},$$

so that g is an algebraic element over $\{A, S\}$. This fact proves our theorem.

Two extensions of the R-module G will be called equivalent, if it is possible to establish between them an isomorphism, by which the elements of G remain fixed.

We prove the following theorem on the algebraic closure of modules:

Theorem 5. *Let R be an arbitrary ring.*

α) Any R-module G has algebraically closed extensions.

β) For any R-module G_0 the following assertions are equivalent:

β_1) G_0 is a maximal algebraic extension of G;

β_2) G_0 is an algebraically closed algebraic extension of G;

β_3) G_0 is a minimal algebraically closed extension of G.

γ) Any R-module G has one, and up to equivalence only one extension G_0, having of the properties β_1), β_2), β_3).[16])

Proof. *Proof of α).* Let τ be a limit-ordinal number, whose cardinal number is greater than the number of elements of R^*. We define the R-modules G_ν for any ordinal ν $(0 \leqq \nu \leqq \tau)$ in the following way:

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[15]) If $A = 0$, then algebraic independence over A coincides with ordinary independence.

[16]) I am indebted to J. Szendrei for kindly having called my attention to the fact that the theorem of Baer [1] and Eckmann and Schopf [4] on the existence and unicity of the minimal injective extension has been generalized recently from the case of unitary modules to the case of arbitrary modules by R. E. Johnson. (Structure theory of faithful rings II. Restricted rings, *Transactions Amer. Math. Soc.*, **84** (1957), 523—544, Theorem 7. 1.)

1. $G_0 = G$.

2. If $\nu - 1$ exists, then let us consider an arbitrary compatible system $[L, \varphi]$ of equations in one unknown over $G_{\nu-1}$. There exists an R-module $G'_{\nu-1}$, in which this system of equations is solvable. By repeated (possibly transfinitely many) uses of this construction we arrive at a module G_ν, which contains $G_{\nu-1}$ as a submodule, and in which every compatible system of equations in one unknown ower $G_{\nu-1}$ is solvable.

3. For limit-ordinals ν the module G_ν is the union of all modules G_μ for $\mu < \nu$.

Now we show that $A = G_\tau$ is an algebraically closed extension of G. Let $[M, \psi]$ be an arbitrary compatible system of equations in one unknown over A. This is a homomorphism ψ into A of the submodule M of the free R-module $R(1)$ (resp. of the left ideal M of the ring R^*). Then there exists an ordinal number σ $(< \tau)$ such that $M\psi \subseteq G_\sigma$. The system of equations $[M, \psi]$ is therefore solvable in $G_{\sigma+1}$ and consequently also in A. Thus A is an algebraically closed module.

Proof of β). Let us suppose that β_1) is fulfilled. Since any algebraic extension of G_0 is an algebraic extension also of G, G_0 has, by our hypothesis, no proper algebraic extension. Thus (by Theorem 3) G_0 is algebraically closed.

Let us suppose that β_2) is fulfilled, and let G_1 be an algebraically closed extension of G, for which $G \subseteq G_1 \subset G_0$. Now this implies $G_0 = G_1 + K$, $K \neq 0$ and so G_0 cannot be an algebraic extension of G. This contradiction proves that G_0 is a minimal algebraically closed extension of G.

Let us suppose that β_3) is fulfilled. Then G_0 has no proper algebraic extension. If, therefore, G_0 is an algebraic extension of G, it must be a maximal algebraic extension of G. Thus we have only to show that G_0 is an algebraic extension of G. Consider an algebraic extension H of G which is maximal in G_0 (Zorn's lemma!). We prove that $H = G_0$. Suppose that H' is an arbitrary algebraic extension of H. Then the identical mapping of H into G_0 can be extended to a homomorphism into G_0 of the whole of H'. Since for the kernel N of this mapping $N \cap H = 0$ holds, we have by the algebraic character of this extension $N = 0$, and so the imbedding of H into G_0 can be extended also to H'. Thus we necessarily have $H' = H$. The module H has therefore no proper algebraic extension, so H is algebraically closed and by our hypothesis $H = G_0$ holds.

Proof of γ). First we show that G has an extension G_0 with property β_2). Let G' be an algebraically closed extension of G (such an extension surely exists in view of α)), and let us consider in G' a maximal algebraic extension H of G. As we have seen in the preceding paragraph, H is algebraically closed, and so $G_0 = H$ is an extension with property β_2) of the module G.

Let us fix G_0, and let $\overline{G}$ be an arbitrary minimal algebraically closed extension of G. Since G_0 is an algebraic extension of G and $G \subseteq \overline{G}$, making use of the method employed in proving the implication $\beta_3) \rightarrow \beta_1$), we can imbed G_0 into $\overline{G}$ so that $\overline{G}$ contains an extension $\overline{G_0}$ of G, equivalent to G_0. Now since G_0 is algebraically closed and $\overline{G}$ is a minimal algebraically closed extension of G, we must have $\overline{G_0} = \overline{G}$. So G_0 and $\overline{G}$ are equivalent minimal algebraically closed extensions of G.

This completes the proof of Theorem 5.

In what follows, we shall make a few additional remarks on algebraically closed modules. It is easy to see that the following statements are valid:

Any direct summand of an algebraically closed module is also algebraically closed.

A submodule of an algebraically closed module is a pure submodule, if and only if it is algebraically closed.

The complete direct sum of algebraically closed modules is also algebraically closed.

Let us now consider the following example:

Let R be the discrete direct sum of the infinitely many rings $S^1, S^2, \ldots$:

$$R = \Sigma S^i,$$

where every ring S^i has at least two elements. Since S^i is an ideal in R, it may be regarded as an R-module. Let us imbed all modules $S^i_{(R)}$ in corresponding algebraically closed R-modules A^i, and consider the direct sum

$$A = \Sigma A^i. \tag{25}$$

We show that A is not an algebraically closed R-module. The set of all equations

$$rx = r \qquad (r \in R)$$

forms evidently a compatible system of equations over A. Nevertheless, this system of equations can have no solution in A, for any element x ($\in A$) has in the direct decomposition (25) only a finite number of nonzero components, and so for any x there exists an index i, such that the equality $r_0 x = r_0$ cannot be valid for any nonzero element $r_0 \in S^i$.

From this example we are able to gather the following facts:

The discrete direct sum of algebraically closed modules is not, in general, algebraically closed.

The union of an ascending chain of algebraically closed modules is not, in general, algebraically closed.

§ 5. A description of the algebraically closed trivial modules.

The following problem seems to be very difficult:

Let R be an arbitrary ring; to determine all algebraically closed R-modules.

In this section we give a modest contribution to this problem, by giving a complete description of the algebraically closed trivial modules.

Let r be an arbitrary element of the ring R, and let us consider the set N of all numbers $m(\in I)$, for which there exists a $t(\in R)$ such that $tr = mr$. N is an ideal in the ring I, and let n be the nonnegative generating element of this ideal. We shall call the number n the *exponent* of the element r, and denote it by $n = E(r)$.

Theorem 6. *For an arbitrary ring R the trivial R-module G with at least two elements is algebraically closed if and only if R has no element of exponent* 0, *and G, as an ordinary abelian group, satisfies the following conditions:*

1. *G is algebraically closed,*

2. *the order of any element of G having positive order, and the exponent of any element of R are relatively prime.*[17])

Proof. Let G be an algebraically closed trivial R-module. If r is an element of exponent 0 of the ring R, then the equation $rx = a(\neq 0, a \in G)$ is compatible, for $\langle t, m\rangle rx = 0$ implies $(tr + mr)x = 0$, $tr = -mr$ and thus $m = 0$, and so $\langle t, m\rangle a = 0$ holds too. The equation $rx = a(\neq 0)$ is however not solvable in G, and consequently R has no element of exponent 0. — Let us consider now the equation

$$nx = c \tag{26}$$

over G with arbitrary $(0 \neq)$ n $(\in I)$ and c $(\in G)$. This is a compatible equation, for the left-hand side of it is being annuled only by elements of the form $\langle r, 0\rangle$ of R^*, and these always annul the right-hand side too. Thus the equation (26) is solvable in G, and so G, as an ordinary abelian group is algebraically closed. — Suppose that the exponent $E(r)$ of some element $r(\in R)$ and the positive order of some element $g(\in G)$ are not relative prime. Then G has an element $g'(\neq 0)$ such that $E(r)g' = 0$. The equation $rx = g'$ is compatible. Indeed, if $\langle t, m\rangle rx = 0$, then $(tr + mr)x = 0$, $tr = -mr$ and consequently $m = m' \cdot E(r)$ for a suitable $m'(\in I)$, and thus

$$\langle t, m\rangle g' = tg' + mg' = mg' = m' \cdot E(r) g' = 0.$$

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17) We shall call element of order 0 what is called in the literature often element of infinite order. We shall denote the order of the element g by $O(g)$.

On the other hand, the equation $rx=g'(\neq 0)$ can have no solution in the trivial R-module G, and, since G is algebraically closed, if $O(g)>0$, then $(E(r), O(g))=1$ for every element $r(\in R)$.

Conversely, let us suppose that R has no element of exponent 0, that G is a trivial R-module and that G, as an ordinary abelian group, satisfies conditions 1) and 2). We show that G is an algebraically closed R-module. Consider the compatible system of equations

$$\langle r_\nu, n_\nu \rangle x = g_\nu (\in G) \tag{27}$$

over G, supposing (of course without loss of generality) that the elements $\langle r_\nu, n_\nu \rangle$ actually run through some left ideal L of R^*. Of course, the elements of the form $\langle r_\mu, 0\rangle$ of the left ideal L also form a left ideal in R^*, and if we consider of the system (27) only the corresponding equations

$$\langle r_\mu, 0\rangle x = g_\mu \tag{28}$$

then we get a system of equations, which, as a subsystem of a compatible system, is clearly also compatible. Let $s_\mu(\in R)$ be an element for which $s_\mu r_\mu = E(r_\mu)\cdot r_\mu$. Then one has

$$\langle s_\mu, -E(r_\mu)\rangle \langle r_\mu, 0\rangle x = 0,$$

and in view of the compatibility of the system (28)

$$\langle s_\mu, -E(r_\mu)\rangle g_\mu = s_\mu g_\mu - E(r_\mu) g_\mu = -E(r_\mu) g_\mu = 0.$$

Since $E(r_\mu)>0$, also $O(g_\mu)>0$ holds, and so in view of $(E(r_\mu), O(g_\mu))=1$ the equation $g_\mu=0$ must hold.

Consider now the system of equations

$$n_\nu x = g_\nu \tag{29}$$

obtained from the system (27). The set of all elements n_ν is an ideal J of the ring I. Let n_{ν_0} be the nonnegative generating element of the ideal J. Then the equation

$$n_{\nu_0} x = g_{\nu_0} \tag{30}$$

has a solution $x=g$ in G, namely if $n_{\nu_0}\neq 0$ then by condition 1), and if $n_{\nu_0}=0$ then by the preceding paragraph $g_{\nu_0}=0$, and so e. g. $g=0$ is clearly a solution. We show that the solution $x=g$ of the equation (30) is a solution also of the system of equations (29). Let indeed be

$$n_\nu x = g_\nu$$

an arbitrary equation of the system (29). Then for a suitable $l(\in I)$ the equation $n_\nu = l n_{\nu_0}$ holds. It is clearly sufficient to show that $g_\nu = l g_{\nu_0}$. Considering the system of equations (27) we obtain

$$\langle r_\nu, n_\nu\rangle x - l\langle r_{\nu_0}, n_{\nu_0}\rangle x = \langle r_\nu - l r_{\nu_0}, n_\nu - l n_{\nu_0}\rangle x =$$
$$= \langle r_\nu - l r_{\nu_0}, 0\rangle x = g_\nu - l g_{\nu_0}$$

and so by the preceding paragraph $g_\nu - lg_{\nu_0} = 0$ i. e. $g_\nu = lg_{\nu_0}$. So the element $x = g(\in G)$ is a solution of (29) and it is also a solution of the system of equations (27), as indeed

$$\langle r_\nu, n_\nu \rangle g = r_\nu g + n_\nu g = n_\nu g = g_\nu$$

holds. This completes the proof of our theorem.

As immediate consequences of Theorem 6 we have

Corollary 1. *If R has an element of exponent 0, then there exists only one algebraically closed trivial R-module, namely the R-module with only one element. If the exponent of any element of R is positive, then all algebraically closed trivial R-modules are given by those trivial R-modules, which, as ordinary abelian groups, are direct sums of groups isomorphic to the additive group of rational numbers and of Prüfer quasicyclic groups which belong to primes relatively prime to the exponents of all elements of the ring R. — In particular, if all elements of R have exponent 1, then any algebraically closed ordinary abelian group is algebraically closed also if taken to be a trivial R-module.*

Corollary 2. *An arbitrary ring R has the property that any algebraically closed ordinary abelian group is algebraically closed also as a trivial R-module, if and only if for any element r of R there exists an element s in R, such that $sr = r$. — In particular, if R is a ring with a left unit element, then any algebraically closed ordinary abelian group is also algebraically closed if considered as a trivial R-module.*

Corollary 3. *If for any element r of the ring R there exists an element s of R such that $sr = r$, then the minimal algebraically closed extension of an arbitrary trivial R-module G can be obtained by taking the minimal algebraically closed extension of G, considered as an ordinary abelian group, and by taking this extension to be a trivial R-module.*

§ 6. Semi-simple rings as operator domains.

In investigating any class of operator-modules, one can raise the question whether there exist, and, in the affirmative case, which are all the rings R, for which any R-module belongs to the class of modules considered. We now raise this question for algebraically closed modules. We have seen in § 5 that for a trivial R-module to be algebraically closed, it is necessary that it be algebraically closed also as an ordinary abelian group. Now, since an abelian group can be considered as a trivial R-module for any ring R, there exist no ring R, for which every R-module is algebraically closed. Therefore we investigate the problem of determining those rings R, for which every R-module is the direct sum of its maximal trivial submodule and of an algebraically closed R-module. The solution of this problem leads to the class of semi-simple rings (in the classical sense).

By a semi-simple ring we mean a ring containing no nonzero nilpotent left ideal and satisfying the descending chain condition for left ideals. According to the well-known Wedderburn-Artin structure theorem such a ring is isomorphic to a direct sum of a finite number of rings, each of which is isomorphic to the complete ring of linear transformations in a suitable finite dimensional vector space over a skew field. By another characterization, *a ring R is semi-simple if and only if every left ideal of R contains a right unit element* (see [6]). Besides this second characterization of semi-simple rings, we make use in our proof also of the following characterization, due to E. NOETHER [16]: *an arbitrary ring is semi-simple if and only if it has a unit element and can be decomposed into the direct sum of minimal left ideals.*

As is shown by the results of our paper [11], the semi-simple rings have interesting properties also as operator domains. Further results in this direction are given by the following theorems:

Theorem 7. *An arbitrary ring R is semi-simple if and only if every R-module G admits a representation in the form of a direct sum*

$$G = G_0 + G_1, \tag{31}$$

where G_0 is the maximal trivial submodule of G and G_1 an algebraically closed R-module.

Theorem 8. *If R is a semi-simple ring, then the compatible system of equations*

$$f_\beta(\ldots, x_\alpha, \ldots) = g_\beta (\in G_1) \tag{32}$$

over the arbitrary unitary R-module G_1 possesses a solution in G_1 and all solutions in G_1 can be obtained by the system of formulae

$$x_\alpha = c_\alpha + \sum_{\delta \in D} d'_{\alpha\delta} h_\delta, \tag{33}$$

where the h_δ are parameters freely chosen from G_1, and the constants c_α ($\in G_1$) are (finite) linear conbinations over R of the elements g_β standing on the right-hand side of the system of equations (32).

Corollary 1. *An arbitrary compatible system of linear equations over a semi-simple ring is solvable in the ring, and all solutions are yielded by the „classical“ system of formulae* (33) [10].

Corollary 2. *If R is a semi-simple ring, then a compatible system of equations over an arbitrary unitary R-module G_1 admits exactly one solution in G_1 if and only if the linear forms on the left-hand sides of the system generate the whole free unitary R-module which is spanned by all unknowns as indeterminates.*

Corollary 3. *If R is a semi-simple ring, then an arbitrary (not necessarily compatible) system of equations over an arbitrary unitary R-module G_1 admits a solution in G_1 if and only if every finite subsystem has a solution in G_1.*

Corollary 4. *If R is a semi-simple ring, then every system of equations over an arbitrary unitary R-module contains a maximal solvable subsystem.*

Corollary 5. *Let R be a semi-simple ring. Then the arbitrary R-module is algebraically closed if and only if its maximal trivial submodule, considered as an ordinary abelian group, is algebraically closed.*

Corollary 6. *Let R be a semi-simple ring and G an arbitrary R-module. Consider the decomposition $G = G_0 \dotplus G_1$ of G into the direct sum of its maximal trivial submodule G_0 and of a unitary module G_1. Let the abelian group A be the minimal algebraically closed extension of G_0, considered as an ordinary abelian group. Take A to be a trivial R-module. Then $A \dotplus G_1$ is the minimal algebraically closed extension of the module G.*

Since Corollaries 1—5 are immediate consequences of Theorems 7 and 8, we prove only Corollary 6. Let $\bar{G}$ be a minimal algebraically closed extension of G. Since, by Theorem 7, G_1 is algebraically closed, a direct decomposition $\bar{G} = \bar{A} \dotplus G_1$ holds with $G_0 \subseteqq \bar{A}$. The module $\bar{A}$, as a direct summand of an algebraically closed module, is algebraically closed, and is evidently a minimal algebraically closed extension of G_0. Now, since R is a ring with unit element, $\bar{A}$ is a trivial R-module, by Corollary 3 of Theorem 6, and considered as an ordinary abelian group, it is a minimal algebraically closed extension of the abelian group G_0.

Proof of Theorems 7 and 8. First let us suppose that for an arbitrary ring R any R-module G is of the form (31), and let us consider accordingly the module $R^*_{(R)}$:

$$R^*_{(R)} = A_0 \dotplus A_1.$$

Let, by this decomposition,

$$\langle 0,1\rangle = e_0 + e_1 \qquad (e_0 \in A_0,\ e_1 \in A_1).$$

If L is an arbitrary left ideal in R, then for each $l (\in L)$ the relation

$$\langle l, 0\rangle = l\langle 0, 1\rangle = le_0 + le_1 = le_1 \in A_1$$

holds, i. e. the set H of the elements $\langle l, 0\rangle$ $(l \in L)$ is a submodule of the module A_1. The maximal trivial submodule of H is 0, so H is algebraically closed. Consider now the, evidently compatible, system of equations

$$\langle l, 0\rangle x = \langle l, 0\rangle \qquad (l \in L)$$

over H, with l running trough all elements of L. Let $\langle e, 0\rangle$ $(e \in L)$ be some solution in H of this system of equations. Then for any $l(\in L)$ the relation $\langle l, 0\rangle\langle e, 0\rangle = \langle l, 0\rangle$ i. e. $le = l$ holds. Thus the element e is the right unit element of the left ideal L. Since L was an arbitrary left ideal of R, the ring R is semi-simple.

Conversely, let R be a semi-simple ring and G an arbitrary R-module. Making use of the Peirce decomposition, we represent G as a direct sum of its maximal trivial submodule G_0 and of a unitary R-module G_1:

$$G = G_0 + G_1.$$

The proof of the remaining part of Theorem 7 as well as the proof of Theorem 8 is based on the following

Lemma 5. *If R is a semi-simple ring, then for every submodule M of the free unitary R-module F generated by the free system of generators x_α $(\alpha \in A)$ there holds a direct representation*

$$F = M + N, \tag{34}$$

where N has the form

$$N = \sum_{\delta \in D} \{s_\delta x_\delta\} \qquad (s_\delta \in R) \tag{35}$$

D being a subset of the index set A.

In order to prove this lemma, let us consider some direct decomposition

$$R = L_1 + L_2 + \cdots + L_m$$

of the semi-simple ring R, where $L_i (i = 1, 2, \ldots, m)$ are minimal left ideals in R, the existence of such a decomposition being assured by the abovementioned theorem of NOETHER. For the unit element 1 of the ring R we accordingly get the representation

$$1 = e_1 + e_2 + \cdots + e_m. \tag{36}$$

By the minimality of L_i we have $Re_i = L_i$ $(i = 1, 2, \ldots, m)$.

Consider now the free unitary R-module F. Making the identification $1 \cdot x_\alpha = x_\alpha$ $(\alpha \in A)$ we may suppose that $x_\alpha \in F$. By the decomposition (36) of the element 1 we have

$$\{x_\alpha\} = \{e_1 x_\alpha\} + \{e_2 x_\alpha\} + \cdots + \{e_m x_\alpha\}, \tag{37}$$

i. e.

$$F = \sum_{\alpha \in A} \{x_\alpha\} = \sum_{\alpha \in A} \sum_{i=1}^{m} \{e_i x_\alpha\}.$$

By Zorn's lemma we select a maximal subset X of the set of all elements $e_i x_\alpha (\alpha \in A, i=1,\ldots,m)$ such that for the submodule $\{X\}$ generated by the set X

$$M \cap \{X\} = 0.$$

We prove the validity of (34) with $N = \{X\}$. For this purpose we have only to show that $e_j x_\lambda \in M + \{X\}$ for all $\lambda \in A$ and $j=1,\ldots,m$, for in this case $x_\lambda \in M + \{X\}$ holds by (37) for any $\lambda (\in A)$. Now by the maximality of the set X we have

$$\{e_j x_\lambda\} \cap (M + \{X\}) \neq 0.$$

But since $\{e_j x_\lambda\}$ is a minimal submodule of F, this implies

$$\{e_j x_\lambda\} \subseteqq (M + \{X\}),$$

i. e. $e_j x_\lambda \in (M + \{X\})$. So we have proved (34) with $N = \{X\}$. As $\{X\}$ is a direct sum of submodules of the form $\{e_i x_\alpha\}$ the representation (35) holds, and, in addition, we see that each s_δ is a sum of some e_i's. (Namely e. g. $\{e_1 x_\delta\} + \{e_2 x_\delta\} = \{(e_1 + e_2) x_\delta\}$ holds.) Thus the proof of the lemma is complete.

In order to conclude the proof of Theorem 7, let $[M, \varphi]$ be an arbitrary compatible system of equations over the unitary R-module G_1. We show that the mapping φ can be extended to an R-homomorphism $\overline{\varphi}$ of the whole module F into $M\varphi$. By virtue of (34) this extension is indeed immediate: for an arbitrary element $f (\in F)$ we have by (34) the unique representation

$$f = f' + f'' \qquad (f' \in M, f'' \in N)$$

and we define

$$f\overline{\varphi} = f'\varphi \qquad (f \in F).$$

This completes the proof of Theorem 7.

Suppose now that the system of equations $[M, \varphi]$ has the „coordinate" form (32). Then the solution $\overline{\varphi}$ considered can be written in the form

$$x_\alpha = c_\alpha \quad (\in M\varphi,\ \alpha \in A). \tag{38}$$

c_α being a linear combination over R of a finite number of the g_β's standing on the right-hand side of the system of equations (32).

Consider now the homogeneous system of equations $[M, \psi]$ corresponding to the system $[M, \varphi]$, i. e. let $M\psi = 0$. In order to get all solutions of this system we construct all possible extensions $\overline{\psi}$ of ψ. In view of the relations (34) and (35) all R-homomorphisms into G_1 of the free unitary module F for which

$$M\overline{\psi} = 0 \tag{39}$$

holds, are determined by the images

$$x_\delta \overline{\psi} = h_\delta \quad (\in G_1, \delta \in D) \tag{40}$$

of the elements x_δ figuring in (35), and, on the other hand, an arbitrary system of prescribed elements $h_\delta(\in G_1, \delta \in D)$ induces by (40), (35), (34) and (39) a well-defined extension $\overline{\psi}$ of ψ. Since, moreover, by (34) and (35) in particular for the elements $x_\alpha(\in F, \alpha \in A)$ the representation

$$x_\alpha = m_\alpha + \sum_{\delta \in D} d_{\alpha\delta}(s_\delta x_\delta) \qquad (m_\alpha \in M)$$

holds, it follows by (40) and (39) that all solutions in G_1 of the homogeneous system of equations $[M, \psi]$ are obtained from the formulae

$$b_\alpha = x_\alpha \overline{\psi} = \sum_{\delta \in D} (d_{\alpha\delta} s_\delta) h_\delta = \sum_{\delta \in D} d'_{\alpha\delta} h_\delta \qquad (\alpha \in A),$$

where the values of the parameters h_δ are to be freely chosen from the elements of the module G_1. Taking now into account that (38) is one of the solutions of the system of equations $[M, \varphi]$, we obtain the solving formulae (33). This completes the proof of Theorem 8.

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